# A Finite Volume Scheme for Calibrating Stochastic Local Volatility Models

*Maarten Wyns (University of Antwerp)* and *Jacques du Toit (Numerical Algorithms Group)*

## Stochastic Local Volatility Model

We consider models of the form

$$dX_\tau = \left(r_d - r_f - \tfrac{1}{2}\sigma^2_{SLV}(X_\tau,\tau)\psi^2(V_\tau)\right)d\tau + \sigma_{SLV}(X_\tau,\tau)\psi(V_\tau)dW^{(1)}_\tau$$
$$dV_\tau = \kappa(\eta - V_\tau)\,d\tau + \xi V^\alpha_\tau dW^{(2)}_\tau$$

Results are presented for the popular **SLV Heston** model ($\alpha = 1/2$) but the method handles other values of $\alpha$ as well.

## Calibrating the Leverage Surface

We must compute a **leverage surface** $\sigma_{SLV}$ consistent with today's prices. This is done by providing a **local volatility surface** $\sigma_{LV}$ and then using Gyöngy's Theorem

$$\sigma^2_{SLV}(x,\tau) = \frac{\sigma^2_{LV}(x,\tau)\displaystyle\int_0^\infty p(x,v,\tau;x_0,v_0)\,dv}{\displaystyle\int_0^\infty \psi^2(v)p(x,v,\tau;x_0,v_0)\,dv}$$

where $p$ is the transition density of the process $(X_t, V_t)_{t\geq 0}$. This density can be obtained by solving the Kolmogorov forward equation

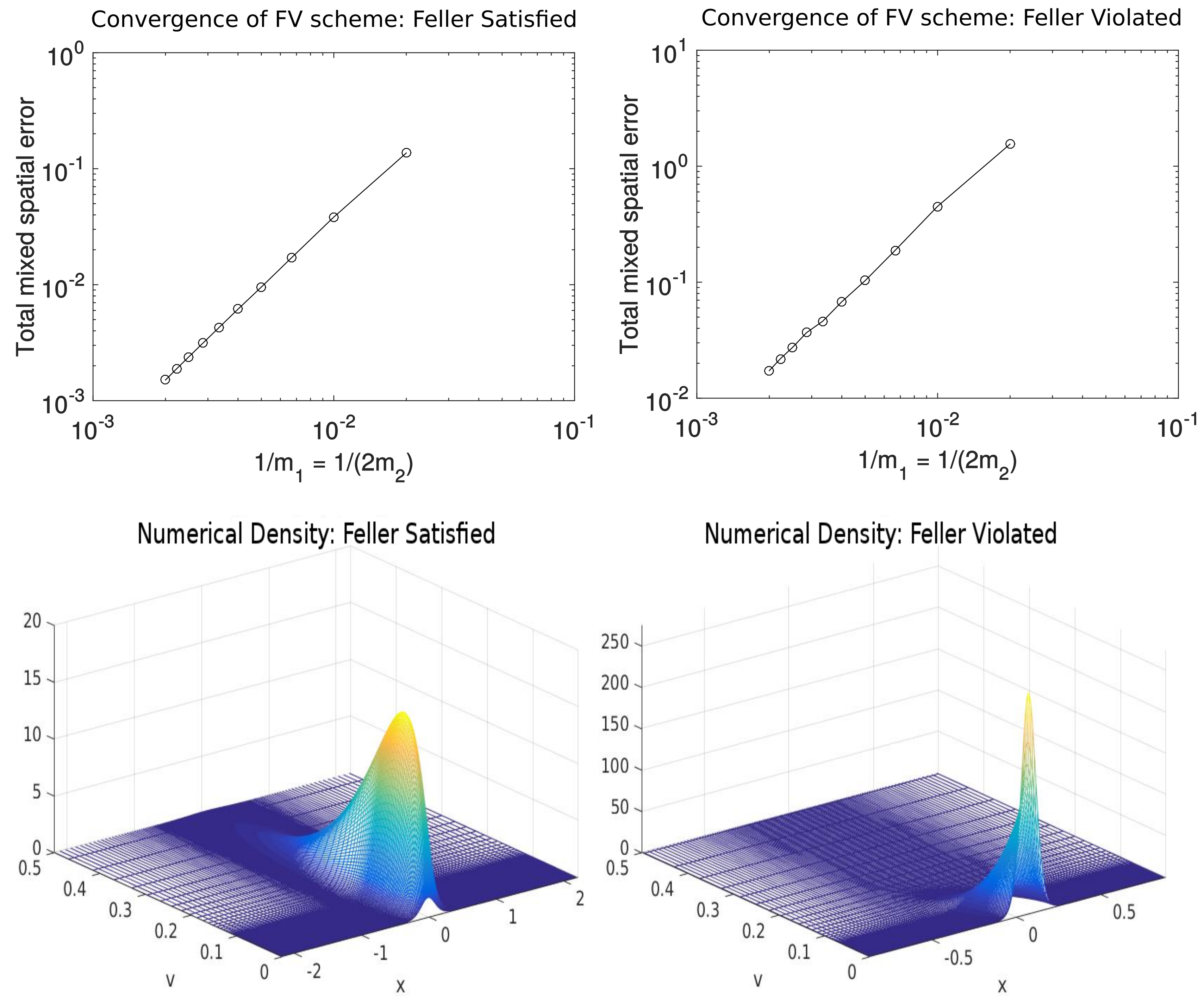
$$\frac{\partial}{\partial\tau}p = \frac{\partial^2}{\partial x^2}\left(\tfrac{1}{2}\sigma^2_{SLV}\psi^2(v)p\right) + \frac{\partial^2}{\partial x\partial v}\left(\rho\xi\sigma_{SLV}\psi(v)v^\alpha p\right) + \frac{\partial^2}{\partial v^2}\left(\tfrac{1}{2}\xi^2 v^{2\alpha}p\right) - \frac{\partial}{\partial x}\left(\left(r_d - r_f - \tfrac{1}{2}\sigma^2_{SLV}\psi^2(v)\right)p\right) - \frac{\partial}{\partial v}\left(\kappa(\eta - v)p\right)$$

## A Second Order ADI Finite Volume Scheme

Solving the forward equation with finite differences is difficult: there are no known boundary conditions at $v = 0$. We apply a standard second order finite volume discretisation to the un–transformed forward equation and impose zero–flux conditions at the boundaries. When $\alpha = 1/2$ we use upwinding at $v = 0$. We smooth the Dirac delta initial condition with 4 Rannacher half steps, whereafter we use Hundsdorfer-Verwer ADI as the time stepping method. We iterate Gyöngy's Theorem twice at each time step: this gives sufficient convergence for $\sigma_{SLV}$.

## Validation Against Heston Model

We took $\alpha = 1/2$ and validated the finite volume method against the known Heston transition density. Convergence and density plots are shown below. We used $m_1$ mesh points in the $x$ direction and $m_2 = m_1/2$ points in the $v$ direction. Convergence is second order when the Feller condition is satisfied, but drops to between first and second order when the Feller condition is violated due to the first order upwinding.

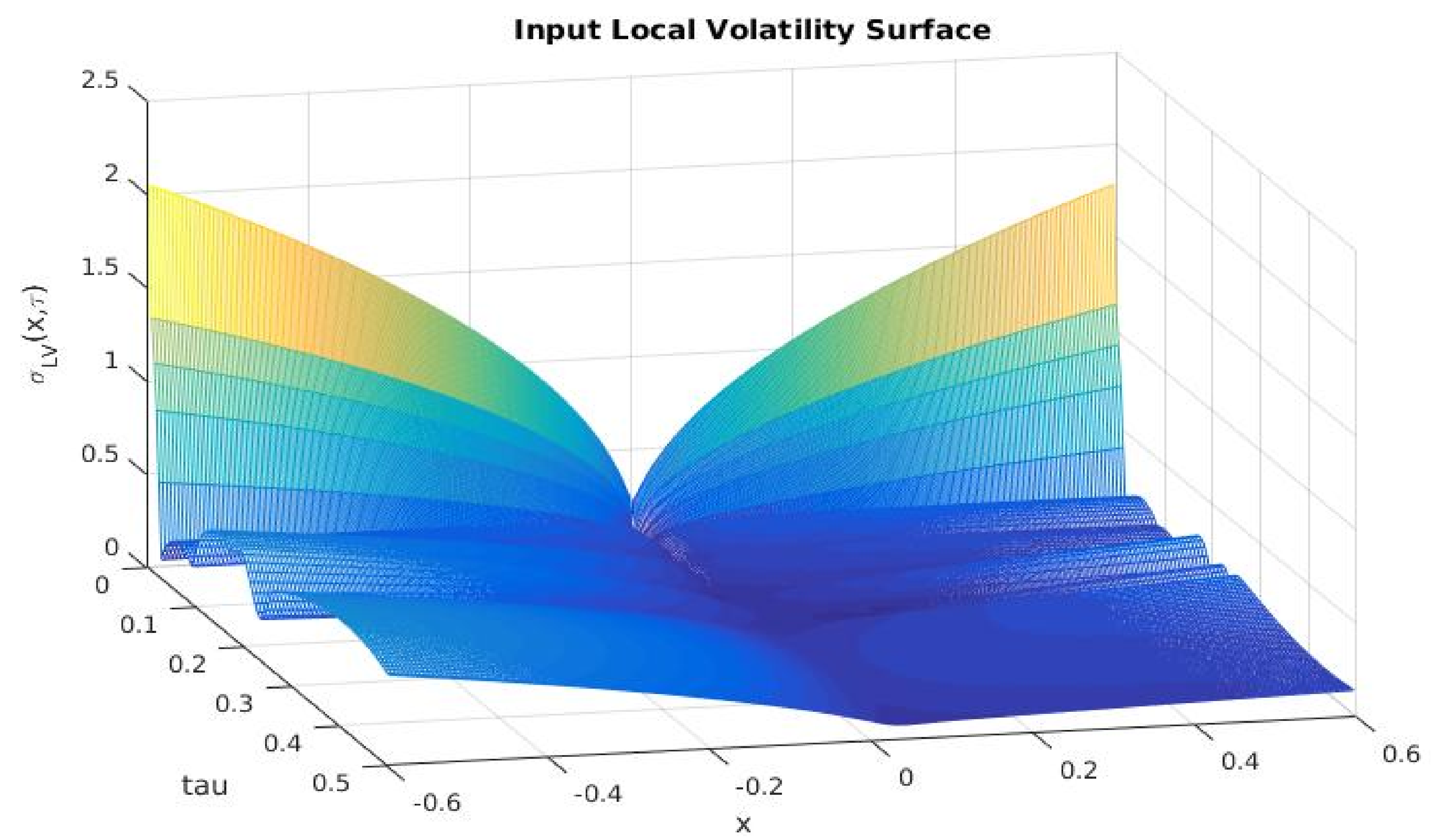



## SLV Calibration on EUR/USD Data from 2 March 2016

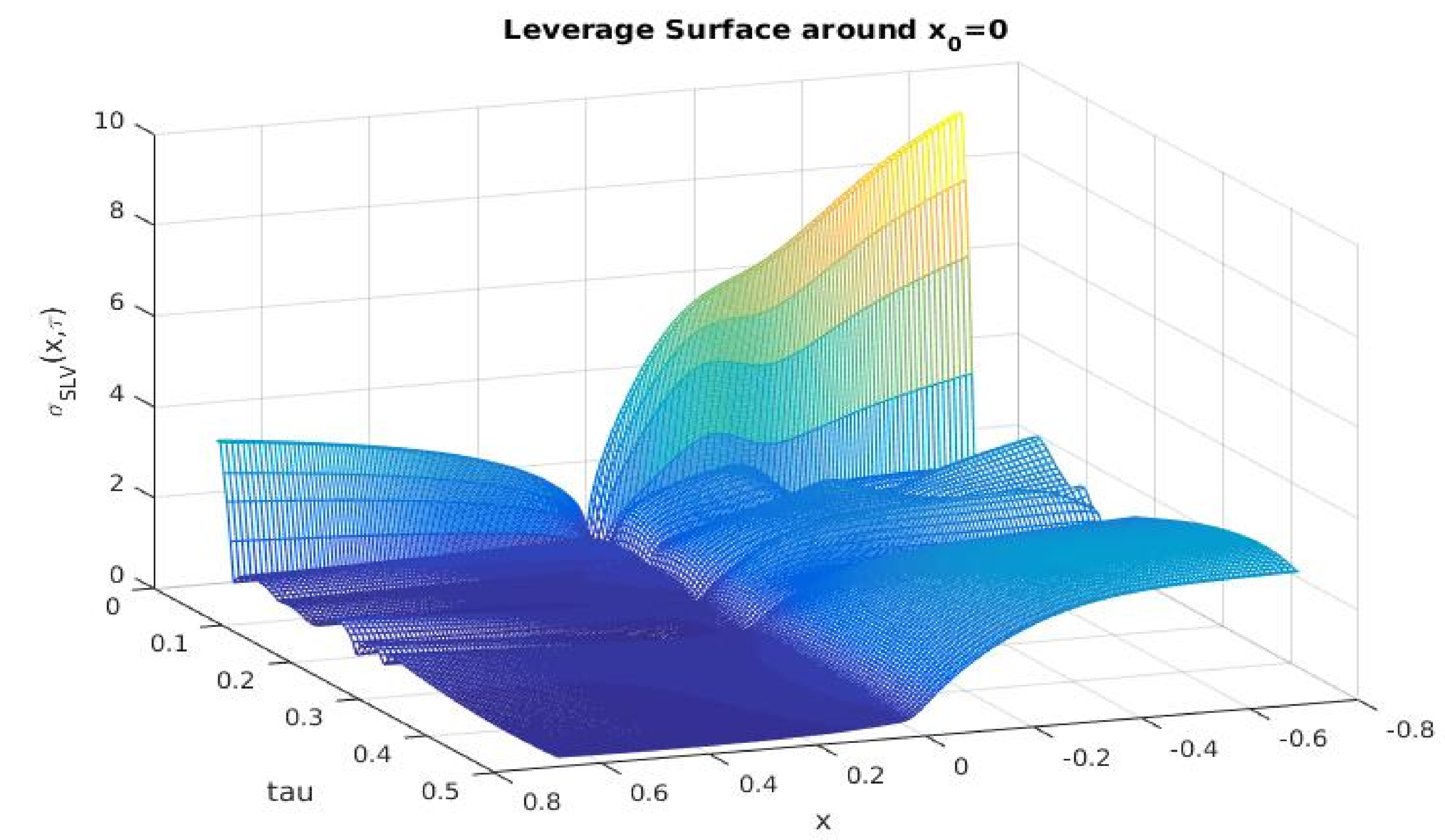
We took market FX quotes and chose challenging stochastic parameters

$$\alpha = 0.5 \quad \kappa = 0.3 \quad \eta = 0.04 \quad \sigma = 0.61 \quad \rho = 0.63 \quad T = 0.5$$

The Feller value is 0.65 indicating strong violation of the Feller condition. A modified SSVI method gave the input local volatility surface:




We calibrated the model using the finite volume ADI method and obtained the following leverage surface:




To evaluate the accuracy of the calibration, we compared the marginal SLV density in $x$ with the density of the local volatility model at $T$. The two agree to within 1e-3. We then compared the implied volatility quotes under the local volatility model to the implied volatilities under the calibrated SLV model. The difference $\varepsilon = |\sigma_{\text{imp},LV} - \sigma_{\text{imp},SLV}|$ is given below.

| $K/S_0$ | 0.75 | 0.80 | 0.90 | 1.0 | 1.10 | 1.20 | 1.25 |
|---|---|---|---|---|---|---|---|
| $\sigma_{\text{imp},LV}$ | 20.48 | 19.10 | 16.16 | 12.50 | 11.52 | 11.93 | 12.30 |
| $\varepsilon$ | 1.3e-2 | 8.9e-3 | 2.9e-3 | 7.4e-4 | 4.3e-3 | 1.0e-2 | 1.5e-2 |

## Implementation and Performance

The Fortran code uses preconditioned GMRES to solve the Rannacher systems. The independent tridiagonal ADI systems are all solved in parallel. On a 12 core, dual socket machine we observe the following performance for calibrating an SLV model on a $600 \times 300$ grid:

| Num Threads | 1 | 2 | 4 | 8 | 12 | 24 | 48 |
|---|---|---|---|---|---|---|---|
| All Rannacher steps | 2.1s | 1.79s | 1.07s | 0.86s | 0.68s | 0.67s | 0.78s |
| One ADI step | 75ms | 59ms | 37ms | 24ms | 18ms | 13ms | 7ms |

## Code Availability

The code with be in the forthcoming version of the NAG Library. Please contact **support@nag.co.uk** for early evaluation options. Further performance and algorithmic improvements are currently being explored.