Algorithmic Differentiation of Nonsmooth and Discontinuous Functions

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Abstract

Adjoint algorithmic differentiation (AAD) is exact up to machine precision and does not capture sensitivity to nearby nonsmoothness or discontinuities. Smoothing the indicator function produces a regularization effect similar to bumping the input while maintaining the efficiency of AAD. The operator-overloading tool dco/c++ supports smoothing through extensible adjoint code patterns for nonsmooth and discontinuous functions.

Adjoint Algorithmic Differentiation

- For $y = f(\mathbf{x})$ with $f : \mathbb{R}^n \to \mathbb{R}$ efficiently compute $\partial f(\mathbf{x})/\partial \mathbf{x}$

AAD	bumping
$O(1) \cdot \mathrm{cost}(f)$	$O(n) \cdot \mathrm{cost}(f)$

- dco/c++ is an operator-overloading AAD tool
  → partial derivatives are stored in tape during computation
  → no separate maintenance of primal and derivative code required

Nonsmooth and Discontinuous Functions

Examples

- Vanilla call payoff (nonsmooth)

$$P(S,K) = \begin{cases} S-K & \text{if } S-K>0 \\ 0 & \text{otherwise .} \end{cases}$$

- Digital payoff (discontinuous)

$$P(S,K) = \begin{cases} 100 & \text{if } S-K>0 \\ 0 & \text{otherwise .} \end{cases}$$

- Nonsmooth and discontinuous functions usually piecewise defined

$$f(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \text{if } g(\mathbf{x})>0 \\ f_2(\mathbf{x}) & \text{otherwise .} \end{cases}$$

- Derivatives near $g(\mathbf{x}) = 0$ can be challenging (e.g. Monte Carlo)

$$\frac{\partial}{\partial \mathbf{x}} \mathbb{E}\left[f(\mathbf{x},\mathbf{z})\right] \not\approx \frac{1}{N}\sum_{i=1}^{N} \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x},\mathbf{z}^i),\ z_1 \ldots z_{n_z} \sim \mathcal{N}(0,1) .$$

- (Stochastic) indicator normal form (INF)

$$f(\mathbf{x}) = \sum_{i=1}^{n_f} \prod_{j=1}^{n_{g,i}} \mathbb{1}\left[g_{i,j}(\mathbf{x},\mathbf{z})>0\right] \cdot f_i(\mathbf{x},\mathbf{z}) .$$

→ Tool-based generation from control flow branching programs

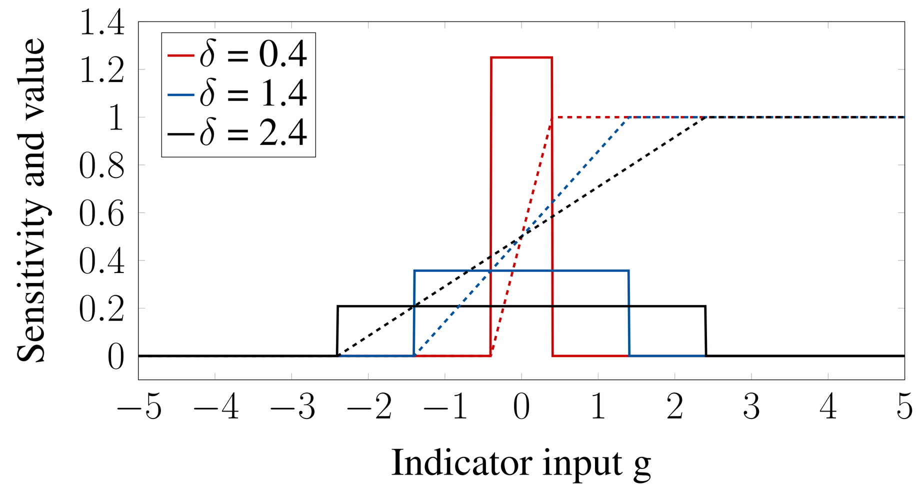
Function Regularization

- Replace indicator by continuous / smooth approximation
- Bandwidth parameter δ controls approximation error (bias)

Uniform Distribution (Call Spread)

$$\mathbb{1}[g>0] \approx \int_{-\infty}^{\infty} \mathbb{1}[g>0] \cdot \mathbb{1}[-\delta<g<\delta]\,\mathrm{d}g$$

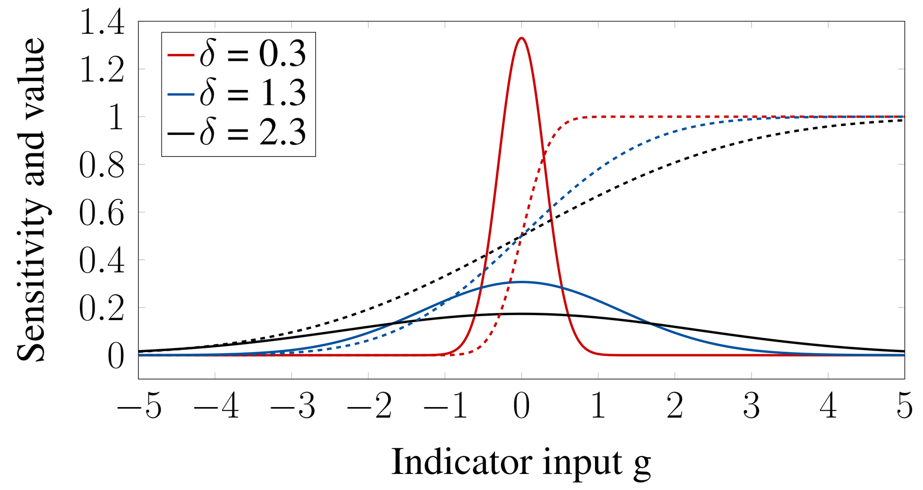
$$\frac{\partial}{\partial g}\mathbb{1}[g>0] \approx \mathbb{1}[-\delta<g<\delta] \cdot \frac{1}{2\delta} .$$



Normal Distribution

$$\mathbb{1}[g>0] \approx \int_{-\infty}^{\infty} \mathbb{1}[g>0] \cdot \frac{1}{\delta\sqrt{2\pi}} \exp\left(-\frac{g^2}{2\delta^2}\right) \mathrm{d}g$$

$$\frac{\partial}{\partial g}\mathbb{1}[g>0] \approx \frac{1}{\delta\sqrt{2\pi}} \exp\left(-\frac{g^2}{2\delta^2}\right) .$$



- Identical tape used to evaluate multiple regularizations
- Tape can evaluate $\partial g_{i,j}(\mathbf{x},\mathbf{z})/\partial \mathbf{z}$ to calibrate bandwidth δ
- Other methods for nonsmooth and discontinuous functions
  → Differentiable univariate quadrature (for stochastic case)
  → Direct evaluation of piecewise linearization

Barrier Option Monte Carlo (Case Study)

- Payoff for discretized path given in stochastic INF

$$P(S_0,K,B,r,\sigma,\mathbf{Z}) = \prod_{i=1}^{N} \mathbb{1}[B-S_i>0] \cdot \mathbb{1}[S_N-K>0] \cdot (S_N-K)$$

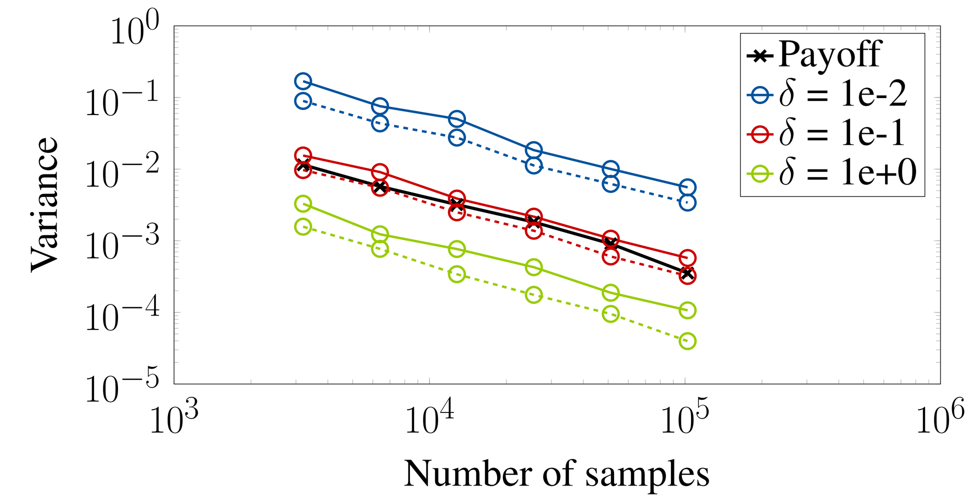
with $S_i = S_i(S_0, r, \sigma, \mathbf{Z})$.

Euler-Maruyama Path with dco/c++

```
dco::ga1s<double>::global_tape->register_variable(S0);
dco::ga1s<double>::global_tape->register_variable(K);
dco::ga1s<double>::global_tape->register_variable(B);
dco::ga1s<double>::global_tape->register_variable(r);
dco::ga1s<double>::global_tape->register_variable(sigma);

S[0] = S0;
for (int i = 1; i < N; ++i) {
  S[i] = S[i-1] + S[i-1]*r*dt + S[i-1]*sigma*sqrt(dt)*Z[i-1];
}

P = 1.0;
for (int i = 0; i < N; ++i) {
  P *= indicator(B - S[i]);
}
P *= indicator(S[N-1] - K) * (S[N-1] - K);

dco::derivative(P) = 1.0;
dco::ga1s<double>::global_tape->interpret_adjoint();

cout << "dPdS0 (Delta) " << dco::derivative(S0) << endl;
cout << "dPdK " << dco::derivative(K) << endl;
cout << "dPdB " << dco::derivative(B) << endl;
cout << "dPdr " << dco::derivative(r) << endl;
cout << "dPdsigma " << dco::derivative(sigma) << endl;
```

Regularized Monte Carlo Delta Estimator

- $S_0 = 100$, $K = 80$, $B = 110$, $r = 0$, $\sigma = 0.2$, $T = 1$
- Variance with different number of samples, $N = 1000$
- Uniform distribution (solid), normal distribution (dashed)



- Delta with $2.048e7$ samples, $N = 1000$

δ	1e-2	1e-1	1e0
uniform	-0.2345	-0.2398	-0.2397
normal	-0.2365	-0.2395	-0.2389

- Low probability events → Variance reduction still necessary

Nearest Correlation Matrix (Case Study)

- Pairwise correlations can be inconsistent
  → Find nearest $n \times n$ correlation matrix that is positive definite
- A_+ projection $\lambda_i := \max(0, \lambda_i)$ of eigenvalues, $A = Q \cdot \mathrm{diag}\,(\lambda) \cdot Q^T$

$$(A_+)_{i,j} = \sum_{k=1}^{n} \mathbb{1}[\lambda_k>0] \cdot \lambda_k \cdot Q_{i,k} \cdot Q_{j,k} .$$

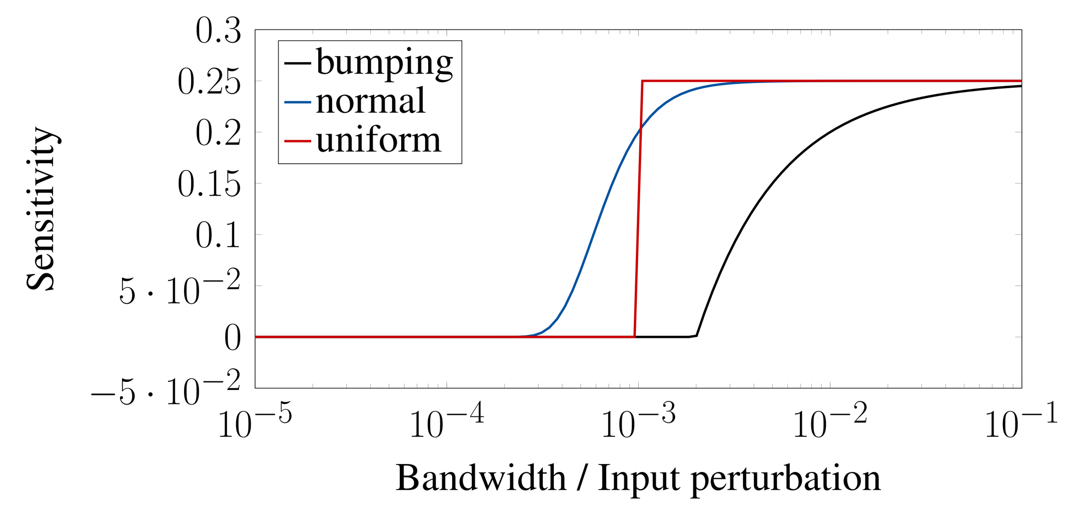
- Convex optimization problem → use implicit function theorem

Regularized Projection

- Test matrix positive definite but numerically singular

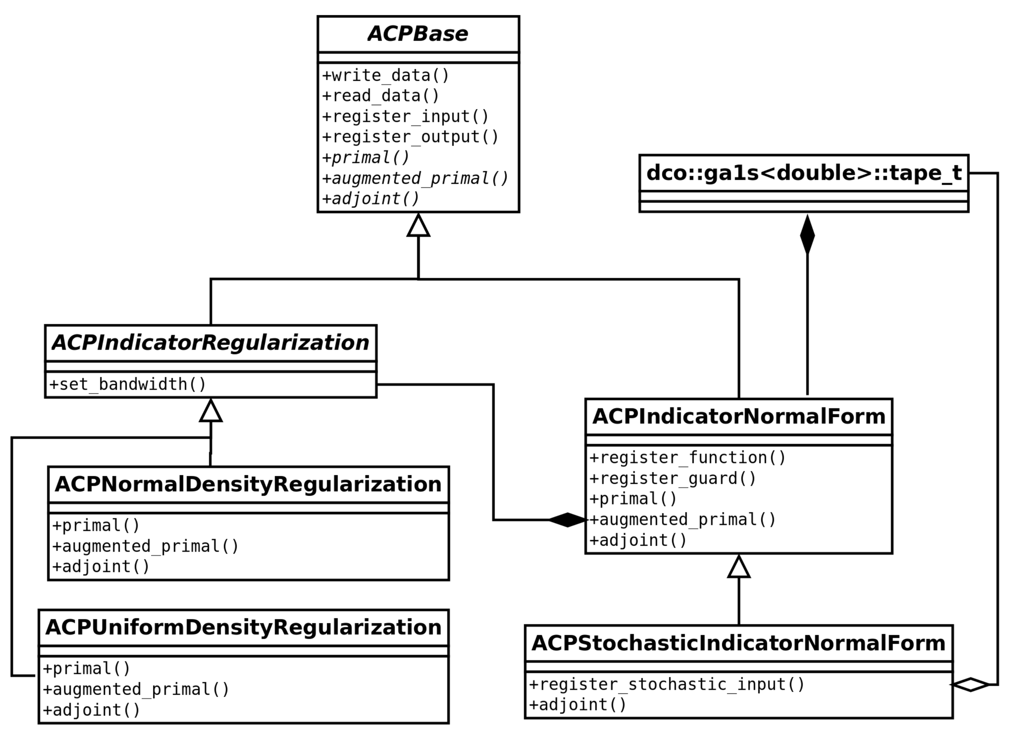
$$A = \begin{pmatrix} 1 & 0.999 & 0 \\ 0.999 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- NCM algorithms maps A to itself, no AAD sensitivities
- Regularization gives sensitivity on implicit function theorem
- Projection sensitivity $\partial(A_+)_{1,1}/\partial A_{2,1}$ for different δ



- Regularization is local bumping (shift)
  → Calibration through correlation uncertainty

Adjoint Code Patterns in dco/c++



Conclusions

- Monte Carlo sensitivities for discontinuous payoffs
- Smoothing of auxiliary functions and implicit function theorem
- Local sensitivity enables bandwidth calibration
- dco/c++ supports extensible adjoint code patterns for regularization

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